Aggregative control of competitive agents with coupled quadratic costs and shared constraints

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Abstract—We consider the problem to control a large population of noncooperative heterogeneous agents, each with strongly convex quadratic cost function depending on the average population state, and all sharing a convex constraint, towards a competitive equilibrium. We assume a minimal information structure through which a central controller can broadcast incentive signals to control the decentralized optimal responses of the agents. We propose a model-free dynamic control law that, based on monotone operator theory arguments, ensures global convergence to an equilibrium independently on the parameters defining the quadratic cost functions, nor on the convex constraints.

I. INTRODUCTION

The problem to coordinate a large population of competitive agents arises in several application domains such as the demand side management in the smart grid [1], [2], [3], [4], e.g. for thermostatically controlled loads [5], [6] and plug-in electric vehicles [7], [8], [9], [10], demand response in competitive markets [11], congestion control for networks of shared resources [12].

The typical challenge in such large-scale control problems is that the agents are noncooperative, self-interested, yet coupled, and have local decision authority that if left uncontrolled can lead to undesired emerging population behavior.

From the control-theoretic perspective, the main objective is to design a control law that can steer the population of agents towards a noncooperative game equilibrium [13].

Whenever the behavior of each agent is affected by some aggregate effect of all the agents, rather than by specific one-to-one effects, which is actually a typical feature of the mentioned application domains, the theory of aggregative games [14], [15], [16], [17] offers the mathematical basics to analyze the strategic interactions between each individual agent and the entire population, although limited to agents with scalar decision variable in the classic literature. For large, in fact infinite, population size, aggregative game setups have been considered as mean field games among agents with quadratic cost functions and unconstrained vector decision variable [20], [21], and in addition with shared constraints.

Generalized games, that is, games among agents with shared constraints [22], [23], have been widely studied in the last decade within the operations research community [24], [25] and the control systems one [26], [27], [28], [29] in relation with duality theory and variational inequalities.

Characterizing the convergence of the dynamic interactions among the noncooperative agents towards an equilibrium is one main question that arises in (generalized) games. With this aim, best response dynamics and fictitious play with inertia, i.e., gradient update dynamics, have been analyzed and designed, respectively, both in discrete [30], [31], [32], [33] and continuous time setups [34], [35]. In particular, fictitious play with inertia has been introduced to overcome the typical non-convergence issue of the best response dynamics [34, Section I]. The common feature of these methods is that the noncooperative agents dynamically implement small gradient steps, each along the direction of optimality relative to the local problem. Thus, the noncooperative agents need to agree on the sequence of step sizes, and cooperate with neighboring players [32], [33], [36], [37], e.g. by exchanging truthful information, in order to update their local directions of optimality. Note that the setup in [33] is more general than that required for generalized aggregative (convex) games.

On the other hand, in this paper we consider aggregative games among noncooperative agents that do not exchange information with the other (competing) agents, nor agree on variables affecting their local behavior. Along the research direction initiated in [20], [21], we instead assume the presence of a central controller ($\kappa$) that coordinates the optimal responses ($x^{1*}, \ldots, x^{N*}$) of the competitive agents, via the broadcast of incentive signals common to all agents. The resulting information structure is semi-decentralized as illustrated in Figure 1.

Specifically, we design a dynamic control law computing incentive vectors that linearly affect all the cost functions, based on the average among the decentralized self-interested optimal responses of the agents. For large population size, these are controlled towards a competitive aggregative equilibrium, that is, a set of agent strategies that are feasible for both the local and the shared constraints, and individually optimal for each agent, given the strategies of all other agents and the penalty associated with the shared constraints.

We aim at a competitive equilibrium rather than a Nash equilibrium since in games with shared constraints the former is more efficient than the latter, in the sense that the (market
clearing) penalty associated with the shared constraints at the Nash equilibrium is higher than that at the competitive equilibrium, as well as the total disutility [11, Theorem 5].

Technically, we establish global convergence of the controlled optimal response dynamics to a competitive equilibrium for large population size, by building upon mathematical tools from variational and convex analysis, and fixed point operator theory [38], [39], independently on the parameters defining the quadratic cost functions are assumed, and coupling constraints are not present.

The paper is organized as follows. Sections II, III define the aggregative game, the competitive equilibrium and its approximation with a fixed point of an appropriate mapping. Section IV presents our dynamic control law and shows its global convergence property. Sections V, VI, VII point at application domains and conclude the paper.

We will refer to [10] for the proofs of (more general counterparts of) the main technical statements in this paper.

Basic notation

\( \mathbb{R}, \mathbb{R}_{\geq 0} \) respectively denote the set of real and positive real numbers; \( \mathbb{N} \) denotes the set of natural numbers; for \( a, b \in \mathbb{N} \), \( a \leq b \), \( \mathbb{N}[a,b] := [a,b] \cap \mathbb{N} \). \( A^T \in \mathbb{R}^{m \times n} \) denotes the transpose of \( A \in \mathbb{R}^{n \times m} \). Given vectors \( x_1, \ldots, x_T \in \mathbb{R}^n \), \( [x_1; \ldots; x_T] \in \mathbb{R}^{nT} \) denotes \( [x_1^T, \ldots, x_T^T]^T \in \mathbb{R}^{nT} \). Given matrices \( A_1, \ldots, A_M \), \( \text{diag}(A_1, \ldots, A_M) \) denotes the block diagonal matrix with \( A_1, \ldots, A_M \) in diagonal positions. With \( \mathbb{S}^n \) we denote the set of symmetric \( n \times n \) matrices; for a given \( Q \in \mathbb{S}^n \), the notations \( Q > 0 \) (\( Q \geq 0 \)) and \( Q \in \mathbb{S}^n_{>0} \) (\( Q \in \mathbb{S}^n_{\geq 0} \)) denote that \( Q \) is symmetric and has positive (non-negative) eigenvalues. \( I \) denotes the identity matrix; \( 0 \) (1) denotes a matrix/vector with all elements equal to 0 (1); to improve clarity, we may add the dimension of these matrices/vectors as subscript. \( A \otimes B \) denotes the Kronecker product between matrices \( A \) and \( B \). Every mentioned set \( S \subseteq \mathbb{R}^n \), \( \{ A \} \subseteq \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \), \( AS+b \) denotes the set \( \{ Ax+b | x \in S \} \); hence given \( S^1, \ldots, S^N \subseteq \mathbb{R}^n \), \( \frac{1}{N} \sum_{i=1}^{N} S^i \) := \( \left\{ \frac{1}{N} \sum_{i=1}^{N} x^i \in \mathbb{R}^n \mid x^i \in S^i \, \forall i \in [1,N] \right\} \). The notation \( \varepsilon = \frac{c}{N} = \mathcal{O}(1/N) \) denotes that there exists \( c > 0 \) such that \( \lim_{N \to \infty} \frac{c}{N} = c \).

Operator theory notation

We denote by \( \mathcal{H}_Q \), with \( Q \in \mathbb{S}_{>0}^n \), the Hilbert space \( \mathbb{R}^n \) with inner product \( \langle \cdot, \cdot \rangle_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) defined as \( \langle x, y \rangle_Q := \langle x^T Q y \rangle \) and induced norm \( \| x \|_Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) defined as \( \| x \|_Q := \sqrt{x^T Q x} \), for all \( x, y \in \mathbb{R}^n \). A mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) is Lipschitz continuous relative to \( \mathcal{H}_Q \) if there exists \( \ell > 0 \) such that \( \| f(x) - f(y) \|_Q \leq \ell \| x - y \|_Q \) for all \( x, y \in \mathbb{R}^n \); \( f \) is a contraction (non-expansive) mapping in \( \mathcal{H}_Q \) if it is Lipschitz relative to \( \mathcal{H}_Q \) with constant \( \ell \in (0,1) \). A mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) is (strictly) monotone in \( \mathcal{H}_Q \) if \( (f(x) - f(y))^T Q (x-y) \geq 0 \) (\( > 0 \)) for all \( x, y \in \mathbb{R}^n \); it is strongly monotone with constant \( \ell > 0 \) in \( \mathcal{H}_Q \) if \( (f(x) - f(y))^T Q (x-y) \geq \ell \| x - y \|_Q^2 \) for all \( x, y \in \mathbb{R}^n \). \( \text{Id} : \mathbb{R}^n \to \mathbb{R}^n \) denotes the identity operator, \( \text{Id}(x) := x \) for all \( x \in \mathbb{R}^n \). Given a closed set \( C \subseteq \mathbb{R}^n \), the projection operator in \( \mathcal{H}_Q \), \( \text{Proj}_Q^C : \mathbb{R}^n \to C \subseteq \mathbb{R}^n \), is defined as \( \text{Proj}_Q^C(x) := \text{arg min}_{y \in C} \| x - y \|_Q = \text{arg min}_{y \in C} \| x - y \|_Q^2 \) for all \( x \in \mathbb{R}^n \).

II. AGGREGATIVE GAME WITH SHARED CONSTRAINTS

We consider a large population of \( N \gg 1 \) competitive agents, where each agent \( i \in [N]\) controls its strategy (i.e., decision variable) \( x^i \in X^i \subset \mathbb{R}^n \), and all share the constraint

\[
\frac{1}{N} \sum_{i=1}^{N} x^i \in S,
\]

for some set \( S \subseteq \mathbb{R}^n \).

We assume that each agent \( i \in [N]\) aims at minimizing its local cost function \( J^i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), which depends on the average on the strategies of all other agents as follows:

\[
x^i_{\text{br}}(x^{-i},p) := \text{arg min}_{y \in X^i} J^i \left( y, \frac{1}{N} (y + \sum_{j \neq i}^{N} x^j) \right) + p^T y,
\]

where the vector \( p \in \mathbb{R}^n \) is the price (i.e., penalty) associated with the shared constraint in (1). Formally, \( x^i_{\text{br}}(x^{-i},p) \) in (2) is the best response of agent \( i \) to the strategies \( x^{-i} := \left( x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^N \right) \) of the other agents, given the penalty vector \( p \).

Throughout the paper, we assume compactness, convexity [21, Section III-A] and Slater’s qualification [41, Proposition 3.3.9] of both the individual and the shared constraints, and strongly convex quadratic cost functions.
Standing Assumption 1: Compactness, convexity, Slater’s qualification. The sets \( X_1, X_2, \ldots, X_N, S \subseteq \sum_{i=1}^{N} X_i \) are compact and convex subsets of \( \mathbb{R}^n \), and satisfy the Slater’s constraint qualification. There exists a compact set \( \mathcal{X} \subseteq \mathbb{R}^n \) such that \( \bigcup_{i=1}^{N} X_i \subseteq \mathcal{X} \) for all \( N \geq 0 \).

Standing Assumption 2: Strongly convex, quadratic cost functions. For all \( i \in \mathbb{N}[1, N] \), the cost function \( J^i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) defined as

\[
J^i(y, \sigma) := \frac{1}{2} y^T Q^i y + c_i^T y + \gamma_i \sigma^T y
\]

(3)

for some \( Q^i \preceq \eta I, c_i \in \mathbb{R}^n, \) and \( \gamma_i \in \mathbb{R} \). □

Note that Standing Assumption 2 implies that \( Q^i \succ 0 \) for all \( i \in \mathbb{N}[1, N] \), and in turn uniqueness of the associated parametric minimizer. Also note that the uniform bounds on the matrices \( \{Q^i\}_{i=1}^{N} \) are postulated as later on we will consider the number \( N \) of agents growing unbounded.

In this paper, we consider the problem to control the strategies of the agents within a competitive aggregative equilibrium, that is, a set of strategies and penalty vector such that: the coupling constraint is satisfied, and each agent’s strategy is optimal given the penalty vector and the strategies of all other agents, see [11, Definition 1] for a definition of competitive equilibrium with linear shared constraints.

Definition 1: Competitive aggregative equilibrium. A pair \( (\bar{x}^i)_{i=1}^{N}, \bar{p} \) is a competitive aggregative \( \varepsilon \)-equilibrium for the game in (2) with shared constraint in (1) if

\[
\frac{1}{N} \sum_{i=1}^{N} \bar{x}^i \in S,
\]

and for all \( i \in \mathbb{N}[1, N], \bar{x}^i \in \mathcal{X}^i \) and

\[
J^i\left(\frac{1}{N} \sum_{i=1}^{N} \bar{x}^i, \frac{1}{N} \sum_{i=1}^{N} \bar{x}^i \right) + \bar{p}^T \bar{x}^i \leq \inf_{y \in \mathcal{X}^i} J^i\left(y, \frac{1}{N} \left( y + \sum_{j \neq i}^{N} \bar{x}^j \right) \right) + \bar{p}^T \bar{y} + \varepsilon.
\]

(4)

It is a competitive aggregative equilibrium for the game in (2) with shared constraint in (1) if \( \frac{1}{N} \sum_{i=1}^{N} \bar{x}^i \in S \) and, for all \( i \in \mathbb{N}[1, N], \bar{x}^i \in \mathcal{X}^i \) and (4) holds with \( \varepsilon = 0 \). □

Proposition 1: Existence of a competitive aggregative equilibrium. There exists a competitive aggregative equilibrium for the game in (2) with shared constraint in (1). □

Proof: See [10].

III. FIXED POINT OF THE AGGREGATION MAPPING

Differently from the usual game theoretic setups, we assume that an agent \( i \) cannot exchange information on the strategies of all other (competitive) agents, which challenges the computation of an equilibrium for large population sizes. Instead, we assume that each agent individually responds optimally to incentive commands \( (\sigma, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \) which enter as second argument of the cost functions in (3) and as penalty vector, respectively. Formally, for all \( i \in \mathbb{N}[1, N] \), we define the optimal response mapping \( x^{i*} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) as

\[
x^{i*}(\sigma, \lambda) := \arg \min_{y \in \mathcal{X}^i} J^i(y, \sigma) + k \lambda^T y
\]

(5)

where \( k \in \mathbb{R} \) is a scalar gain to be designed.

Remark 1: For the sake of simplicity, in (5) we have considered scalar parameters \( \gamma, \lambda \in \mathbb{R} \). More generally, these parameters can be replaced by matrices \( C, K \in \mathbb{R}^{n \times n} \), respectively, by making the analysis more involved [10]. □

We then define the (augmented) aggregation mapping \( A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) as

\[
A(\sigma, \lambda) := \frac{1}{N} \sum_{i=1}^{N} x^{i*}(\sigma, \lambda)
\]

(6)

and note that, if \( \bar{\sigma} = A(\bar{\sigma}, \bar{\lambda}) \) for some \( \bar{\lambda} \in \mathbb{R}^n \), and we introduce the shorthand notation \( \bar{x}^i := x^{i*}(\bar{\sigma}, \bar{\lambda}) \), then \( \bar{\sigma} = \frac{1}{N} \sum_{i=1}^{N} \bar{x}^i \) is the second argument of \( J^i \) in the left-hand side of (4), and an \( O(1/N) \) approximation of the second argument of \( J^i \) in the right-hand side of (4). Thus, let us formalize next that such a pair \( (\bar{x}^i)_{i=1}^{N}, \bar{\lambda} \) is in fact a competitive aggregative \( \varepsilon \)-equilibrium.

Proposition 2: Existence of a fixed point. There exists \( (\bar{\sigma}, \bar{\lambda}) \in (S \times \mathbb{R}^n) \) such that \( \bar{\sigma} \in \frac{1}{N} \sum_{i=1}^{N} \mathcal{X}^i \) is a fixed point of \( A(\cdot, \lambda) \) in (6), i.e., \( \bar{\sigma} = A(\bar{\sigma}, \bar{\lambda}) \). □

Proof: See [10].

IV. CONTROL OF DECENTRALIZED OPTIMAL RESPONSES TOWARDS A COMPETITIVE AGGREGATIVE EQUILIBRIUM

With the aim to control the decentralized optimal responses towards a competitive aggregative equilibrium for large population size, that according to Theorem 1 is generated by a fixed point of \( A(\cdot, \lambda) \) in \( S \) for an appropriate \( \lambda \in \mathbb{R}^n \), in this section we make use of the mapping \( x^{0*} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) defined as

\[
x^{0*}(\sigma, \lambda) := \arg \min_{y \in \mathcal{S}} \frac{1}{2} y^T y + k \sigma^T y - k \lambda^T y,
\]

(7)

so that a pair \( (\bar{\sigma}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^n \) satisfies \( \bar{\sigma} = A(\bar{\sigma}, \bar{\lambda}) = x^{0*}(\bar{\sigma}, \bar{\lambda}) \in S \) if and only if \( (\bar{\sigma}, \bar{\lambda}) \in \mathbb{R}^{2n} \) is a zero of the mapping \( \Theta : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) defined as

\[
\Theta\left(\begin{array}{c}
\sigma \\
\lambda
\end{array}\right) := \left[\begin{array}{c}
\sigma - A(\sigma, \lambda) \\
\sigma - 2A(\sigma, \lambda) + x^{0*}(\sigma, \lambda)
\end{array}\right],
\]

(8)
i.e., $\Theta([\sigma; \lambda]) = 0_{2n}$.

In general, computing a zero of a multi-variable nonlinear mapping is a challenging task. However, for monotone mappings there exist iterative algorithms with global convergence guarantees [38, Chapter 25]. In the following, we show that by choosing the gain $k$ in (5) large enough the mapping $\Theta$ in (8) is monotone in some Hilbert space. Consequently, we derive a dynamic control law that ensures global convergence of the controlled decentralized optimal responses to a competitive aggregative $\varepsilon$-equilibrium, with $\varepsilon = \varepsilon_N = \mathcal{O}(1/N)$.

**Design choice 1: Sufficiently large gain.** The gain $k$ in (5) is chosen such that $k > \max \{0, -\gamma\}$. □

**Theorem 2: Monotonicity.** Under design choice 1, the mapping $\Theta$ in (8) is monotone in $\mathcal{H}_P$, with

$$P := \begin{bmatrix} (\gamma + 2k) & -k \\ -k & k \end{bmatrix} \otimes I > 0. \quad (9)$$

**Proof:** See [10]. □

To compute a zero of $\Theta$ in (8), we propose the following dynamic controller $\kappa$, inspired by the Tseng’s zero-finding algorithm [38, Equation 25.48, Remark 25.11]:

$$\begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} \leftarrow \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} - \alpha_t \Theta \left( \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} - \alpha_t \Theta \left( \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} \right) \right), \quad (10)$$

where the sequence $(\alpha_t)_{t=0}^\infty$ satisfies the following condition.

**Design choice 2: Sufficiently small stepsizes.** The sequence $(\alpha_t)_{t=0}^\infty$ is chosen such that

$$\alpha_t := \tilde{\alpha} \in (0, \ell_{\Theta}^{-1}) \quad \forall t \geq \tilde{t}, \quad (11)$$

for some $\tilde{t} \in \mathbb{N}$, where $\ell_{\Theta} > 0$ is the Lipschitz constant of $\Theta$ in (8) relative to $\mathcal{H}_P$, with $P$ as in (9). □

Note that the mapping $\Theta$ in (8) can be written as the composition of affine and projection mappings. Thus $\Theta$ is globally Lipschitz continuous with constant $\ell_\Theta$, which can be explicitly upper bounded as a function of the problem data $q$, $\gamma$, $\sigma$, and the choice of the gain $k$. For the interest of space, we omit the explicit computation of such an upper bound.

**Theorem 3: Decentralized convergence.** If Assumptions 1, 2 hold, then the sequence $(\sigma(t), \lambda(t))_{t=0}^\infty$ defined iteratively in (10) converges, for any initial condition $(\sigma(0), \lambda(0)) \in \mathbb{R}^n \times \mathbb{R}^n$, to a zero of $\Theta$ in (8), where $\mathcal{A}$ is as in (6) and $x^{\star}$ is as in (5) for all $i \in \mathbb{I}[1, N]$. □

**Proof:** We verify the assumptions of Tseng’s theorem [38, Theorem 25.10 (A = 0 $\Rightarrow$ J$\gamma$A = Id), Remark 25.11].

The mapping $\Theta(\cdot) := \left[ \begin{array}{cc} I_n \otimes 0_{\alpha_{\text{max}}} \\ I_n \otimes 0_{\alpha_{\text{max}}} \end{array} \right] \cdot \left( -\frac{1}{N} \sum_{i=1}^N \Gamma^i \right)$ in (8) is single valued, Lipschitz continuous and monotone in $\mathcal{H}_P$, $P$ in (9), by Theorem 2, due to design choice 1. Therefore, Tseng’s algorithm [38, Equation 25.48] is guaranteed to converge, for any initial condition, if the asymptotic stepsize $\tilde{\alpha}$ in (11) is constant and less or equal than the inverse of the Lipschitz constant $\ell_\Theta$ of the mapping $\Theta$, relative to $\mathcal{H}_P$. □

**Corollary 1: Decentralized convergence.** Under design choices 1, 2, the sequence

$$\left( (x^i, (\sigma(t), \lambda(t)))_{t=1}^N, k\lambda(t) \right)_{t=0}^\infty$$

defined iteratively from (10) converges, for any initial condition $(\sigma(0), \lambda(0)) \in \mathbb{R}^n \times \mathbb{R}^n$, to a competitive aggregative $\varepsilon_N$-equilibrium of the game in (2) with shared constraint in (1), with $\varepsilon_N = \mathcal{O}(1/N)$. □

**Proof:** It follows from Theorems 1, 3. □

The main feature of the dynamic control algorithm in (10) is that the iteration consists in two steps, and in fact it is called forward-backward-forward algorithm [38, Equation 25.48].

One computational advantage of the iteration in (10) is that it only requires one-to-all coordination between a central controller and the decentralized, hence parallelizable, optimal responses $(x^i)_{i=1}^N$ in (5) of the agents, as illustrated in Algorithm 1. We also note that at each iteration $t$ in (10) only two vectors in $\mathbb{R}^n$ are broadcast, independently on the population size $N$, which can be arbitrarily large. In addition, the central controller needs access to the aggregate information $A(\sigma(t), \lambda(t)) \in \mathbb{R}^n$ only, not necessarily to the entire set of optimal responses.

On the other hand, one potential computational drawback is that no convergence rate is known in general for the forward-backward-forward algorithm [38, Chapter 25].

**V. Applicability**

The generalized aggregative game setup in (1)–(2) is applicable to demand side management for large populations of prosumers in the smart grid, including residential loads with coupling constraints [2], [42], smart homes with shared renewable energy sources [43] and plug-in electric vehicles with transmission line constraints [44], [45], demand response in competitive markets [11], as well as congestion control with network capacity constraints [12].

The common feature of these applications is in fact the presence of a population of competitive agents with strongly convex (almost) quadratic cost functions, coupled together through the average among the strategies of all agents, and convex local and shared constraints.

Most importantly, in all the mentioned applications bidirectional peer-to-peer communication is not available and/or too costly. Therefore, distributed algorithms that require peer-to-peer communication among agents are currently not implementable. On the other hand, semi-decentralized control architectures are much less expensive from the communication point of view.

We refer to [10] for illustrative numerical simulations relative to some of the mentioned application domains.
Algorithm 1: Aggregative control of competitive optimal responses

Initialization: \( t \leftarrow 0 \);
- The controller chooses \( \begin{bmatrix} \sigma(0) \\ \lambda(0) \end{bmatrix} \).

Iterate until convergence:
- The controller broadcasts
  \[
  \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix}
  \]
  and computes \( x^0 * (\sigma(t), \lambda(t)) \) in (7).
- The agents compute in parallel \( x^* (\sigma(t), \lambda(t)) \) in (5), for all \( i \in \mathbb{N}[1, N] \).
- The controller receives \( A(\sigma(t), \lambda(t)) \) in (6), computes \( \Theta(\sigma(t), \lambda(t)) \) in (8), transmits
  \[
  \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} - \alpha_t \Theta \left( \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} \right) =: \begin{bmatrix} \sigma(t+1) \\ \lambda(t+1) \end{bmatrix}
  \]
  and computes \( x^0 * (\sigma(t+1), \lambda(t+1)) \) in (7).
- The agents compute in parallel \( x^* (\sigma(t+1), \lambda(t+1)) \), for all \( i \in \mathbb{N}[1, N] \).
- The controller receives \( A(\sigma(t+1), \lambda(t+1)) \) in (6), computes \( \Theta(\sigma(t+1), \lambda(t+1)) \) in (8) and broadcasts
  \[
  \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} - \alpha_t \Theta \left( \begin{bmatrix} \sigma(t+1) \\ \lambda(t+1) \end{bmatrix} \right) =: \begin{bmatrix} \sigma(t+1) \\ \lambda(t+1) \end{bmatrix}.
  \]
- \( t \leftarrow t + 1 \).

VI. NUMERICAL EXAMPLE

In this section, we numerically simulate Algorithm 1 on an academic example. We consider \( N \) agents with: local constraints \( x^i \in \mathcal{X}^i := [-1, 1]^n \cap \left\{ y \in \mathbb{R}^n \mid F^i y \leq 0 \right\} \), where \( F^i \in [-1, 1]^n \) is randomly chosen; cost functions \( f^i(x^i, \sigma) := \|x^i - \hat{x}^i\|^2 Q_i + 2\gamma \sigma^\top x^i \), where \( \hat{x}^i \in [-1, 1]^n \) and \( Q_i \succ 0 \) are randomly chosen; coupling constraint \( \frac{1}{N} \sum_{i=1}^N x^i \in \mathcal{S} := [\frac{1}{2}, 1]^n \). We choose \( n = 3 \), \( \gamma = -1 \) and design \( k = 2 \) according to design choice 1, and \( \alpha_k := \max \left\{ \frac{1}{2\kappa + 1}, \alpha \right\} \), for sufficiently small \( \alpha > 0 \), according to design choice 2.

We compare the convergence behavior induced by the (two-step) dynamic control law \( \kappa \) in (10) with that by the (one-step) simplified control law
(12)
\[
\begin{bmatrix} \sigma(t+1) \\ \lambda(t+1) \end{bmatrix} = (\text{Id} - \alpha_t \Theta) \left( \begin{bmatrix} \sigma(t) \\ \lambda(t) \end{bmatrix} \right).
\]
Note that the latter has no global convergence guarantee according to the technical results in this paper.

Figures 2, 3 show the typical convergence pattern (for \( \alpha = 0.1 \)) towards a zero of the mapping \( \Theta \). In our numerical experience, we always notice that the simplified scheme in (12) has faster convergence than the standard one. This motivates the theoretical investigation of conditions under which global convergence can be guaranteed via a one-step dynamic control law.

VII. CONCLUSION

We have addressed the problem to control a large population of competitive agents, with strongly convex quadratic cost functions coupled together via the average population state, convex local and coupling constraints, towards a competitive aggregative equilibrium. Our results generalize that in [21] to handle the presence of shared constraints, and allow us to design of a model-free dynamic control law with global convergence guarantee. Future work will focus on the convergence rate analysis and on addressing a more general setup with non-convex convex cost functions.

The connection with cooperative optimization setups with
shared constraints [46, 47] for large population size is also to be studied.

REFERENCES